Solution Key to re-exam in Financial Econometrics A: Volatility Modelling, February 2016

Question A:

Consider the ARCH model given by,

$$x_t = \sigma_t \eta_t, \qquad t = 1, 2, ..., T \tag{1}$$

with η_t i.i.d.N$(0, 1)$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta z_{t-1}^2. \tag{2}$$

Here z_t is some exogenous covariate, as for example the realized volatility.

Question A.1: Suppose that $\beta = 0$ and recall that $E\eta_t^4 = 3$. Derive a condition for x_t to be weakly mixing and such that $Ex_t^4 < \infty$.

Solution: For $\beta = 0$ the transition density of x_t is given by $f(x_t|x_{t-1}) = \frac{1}{\sqrt{2\pi(\omega+\alpha x_{t-1}^2)}} \exp\left(-\frac{x_t^2}{2(\omega+\alpha x_{t-1}^2)}\right)$ which is strictly positive (provided that $\omega > 0$) and continuous in x_t and x_{t-1}. This enables us to establish the drift criterion for x_t. In order to ensure that $Ex_t^4 < \infty$, we choose the drift function $\delta(x) = 1 + x^4$. Standard derivations from the lectures yield that $\alpha < 1/\sqrt{3}$ is a sufficient condition for x_t being weakly mixing with $Ex_t^4 < \infty$.

Question A.2: Now consider the case of $\beta > 0, \omega > 0$ and $\alpha \geq 0$. Assume that also z_t is i.i.d.N$(0, \sigma_z^2)$, and that z_t and η_t are independent. Define the bivariate vector $v_t = (x_t, z_t)'$ and observe that the density of v_t conditional on v_{t-1} is given by,

$$f\left(v_t|v_{t-1}\right) = \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_z^2 \sigma_t^2}} \exp\left(-\frac{1}{2}\left\{\frac{x_t^2}{\sigma_t^2} + \frac{z_t^2}{\sigma_z^2}\right\}\right). \tag{3}$$

Argue that v_t is a Markov chain for which the transition density $f\left(\cdot|\cdot\right)$ is such that the drift criterion can be applied.

Next, with drift function $\delta\left(v_t\right) = 1 + v_t' v_t = 1 + x_t^2 + z_t^2$ and $v = (x, z)'$, show that for some constant c

$$E\left(\delta\left(v_t\right)|v_{t-1} = v\right) \leq c + \max\left(\alpha, \beta\right)\left(x^2 + z^2\right). \tag{4}$$

Conclude that if $\max\left(\alpha, \beta\right) < 1$, then v_t is weakly mixing with $E\left\|v_t\right\|^2 \leq E[x_t^2] + E[z_t^2] < \infty$.

Solution: The Markov chain has a nice transition density due to the fact that $f(v_t|v_{t-1})$ is strictly positive and continuous in v_t and v_{t-1}. Next,

$$\begin{aligned}
E\left(\delta\left(v_t\right)|v_{t-1}=v\right) &= E\left(1+x_t^2+z_t^2|v_{t-1}=v\right) \\
&= 1+E\left(x_t^2|\left(x_{t-1},z_{t-1}\right)'=(x,z)'\right)+E\left(z_t^2|\left(x_{t-1},z_{t-1}\right)'=(x,z)'\right) \\
&= 1+E\left(\sigma_t^2\eta_t^2|\left(x_{t-1},z_{t-1}\right)'=(x,z)'\right)+E\left(z_t^2\right) \\
&= 1+\omega+\alpha x^2+\beta z^2+\sigma_z^2 \\
&\leq 1+\omega+\sigma_z^2+\max(\alpha,\beta)(x^2+z^2) \\
&= c+\max(\alpha,\beta)v'v.
\end{aligned}$$

By the usual arguments the drift criterion is satisfied if $\max(\alpha,\beta)<1$.

Question A.3: With $L_T\left(\omega,\alpha,\beta\right)$ the log-likelihood function for the ARCH model, it holds that the score for β is given by,

$$S\left(\omega,\alpha,\beta\right)=\partial\log L_T\left(\omega,\alpha,\beta\right)/\partial\beta=\sum_{t=1}^{T}\frac{1}{2}\left(\frac{x_t^2}{\sigma_t^2}-1\right)\frac{z_{t-1}^2}{\sigma_t^2}. \tag{5}$$

Show that with $\omega_0>0$, $\alpha_0<1$ and $0<\beta_L\leq\beta_0<1$ then under the condition that $v_t=(x_t,z_t)'$ is weakly mixing,

$$\frac{1}{\sqrt{T}}S\left(\omega_0,\alpha_0,\beta_0\right)\xrightarrow{d}N\left(0,\frac{\nu}{2}\right), \tag{6}$$

where $\nu=E[(z_{t-1}^2/\left(\omega_0+\alpha_0x_{t-1}^2+\beta_0z_{t-1}^2\right))^2]\leq 1/\beta_L^2<\infty$.

Solution: The asymptotic normality of $T^{-1/2}S\left(\omega_0,\alpha_0,\beta_0\right)$ is established using the CLT for martingale differences from the lecture notes. Evaluated at $\theta_0=\left(\omega_0,\alpha_0,\beta_0\right)'$,

$$\frac{1}{2}\left(\frac{x_t^2}{\sigma_t^2(\theta_0)}-1\right)\frac{z_{t-1}^2}{\sigma_t^2(\theta_0)}=\frac{1}{2}(\eta_t^2-1)\frac{z_{t-1}^2}{\omega_0+\alpha_0x_{t-1}^2+\beta_0z_{t-1}^2}=f(v_t,v_{t-1}),$$

so $S\left(\omega_0,\alpha_0,\beta_0\right)=\sum_{t=1}^{T}f(v_t,v_{t-1})$. Using that that v_t is weakly mixing. It hence suffices to show that $E[f(v_t,v_{t-1})|v_{t-1}]=0$ and $E[f^2(v_t,v_{t-1})]<\dot{\infty}$. First,

$$\begin{aligned}
E[f(v_t,v_{t-1})|v_{t-1}] &= E\left[\frac{1}{2}(\eta_t^2-1)\frac{z_{t-1}^2}{\omega_0+\alpha_0x_{t-1}^2+\beta_0z_{t-1}^2}|v_{t-1}\right] \\
&= \frac{1}{2}\frac{z_{t-1}^2}{\omega_0+\alpha_0x_{t-1}^2+\beta_0z_{t-1}^2}E\left[(\eta_t^2-1)|v_{t-1}\right] \\
&= \frac{1}{2}\frac{z_{t-1}^2}{\omega_0+\alpha_0x_{t-1}^2+\beta_0z_{t-1}^2}E\left[(\eta_t^2-1)\right] \\
&= 0.
\end{aligned}$$

Next,

$$
\begin{aligned}
E[f^2(v_t, v_{t-1})] &= E\left[\frac{1}{4}(\eta_t^2 - 1)^2 \left(\frac{z_{t-1}^2}{\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 z_{t-1}^2}\right)^2\right] \\
&= \frac{1}{4} E\left[(\eta_t^2 - 1)^2\right] E\left[\left(\frac{z_{t-1}^2}{\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 z_{t-1}^2}\right)^2\right] \\
&= \frac{1}{2} E\left[\left(\frac{z_{t-1}^2}{\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 z_{t-1}^2}\right)^2\right] \\
&\leq \frac{1}{2}\frac{1}{\beta_0^2} \leq \frac{1}{2}\frac{1}{\beta_L^2} < \infty,
\end{aligned}
$$

since $\beta_0 \geq \beta_L > 0$. By the CLT for martingale differences, we conclude that

$$
T^{-1/2} S\left(\omega_0, \alpha_0, \beta_0\right) \overset{D}{\to} N(0, E[f^2(v_t, v_{t-1})]) \quad \text{as } T \to \infty,
$$

where $E[f^2(v_t, v_{t-1})] = \frac{1}{2}\nu$.

Question A.4: With z_t Realized volatility for S&P500 and x_t log-returns on S&P500, ML estimation gave:

Output: MLE of ARCH with RV	
$\hat{\alpha} = 0.11$	std.deviation$(\hat{\alpha}) = 0.012$
$\hat{\beta} = 0.09$	std.deviation$(\hat{\beta}) = 0.091$

What would you conclude in terms of the importance of Realized volatility?

Solution: Based on the estimation output one may conclude (based on the usual critical values) that $\alpha > 0$ whereas one cannot reject that $\beta = 0$. This suggests that the realized volatility is not an important exogenous variable in the volatility equation σ_t^2. The very good answer might relate this to Question A.3 where it was used that $\beta_0 > 0$ in order to show that the variance of the score, i.e. $E[f^2(v_t, v_{t-1})]$, is finite. When $\beta_0 = 0$ we probably need a condtion such as $E z_{t-1}^4 < \infty$. So in order to test whether $\beta = 0$ might require stronger conditions on v_t.

Question B:

Consider the switching-ARCH(1) model given by

$$y_t = \sigma_t z_t$$
$$\sigma_t^2 = \omega_0 + \omega_1 1_{(S_t=1)} + \alpha y_{t-1}^2$$

where z_t and S_t are independent, with z_t i.i.d.N$(0,1)$ and S_t can take value 1 or 2. Note that $1_{(S_t=1)} = 1$ if $S_t = 1$ and $1_{(S_t=1)} = 0$ if $S_t = 2$. Moreover, $\omega_0 > 0$, $\omega_1 \geq 0$, and $\alpha \geq 0$.

Question B.1: Suppose that $\alpha = \omega_1 = 0$. Explain if y_t is weakly mixing.

Solution: When $\alpha = \omega_1 = 0$, $y_t = \omega_0^{1/2} z_t$ meaning that y_t is i.i.d. and hence weakly mixing.

Question B.2: Next, assume that S_t is a Markov chain evolving according to the transition probabilities $p_{ij} = P(S_t = j | S_{t-1} = i)$, $i, j = 1, 2$ where the transition probabilities p_{ij} are such that S_t is weakly mixing.

Suppose that $\alpha = 0$ while $\omega_1 > 0$. Explain if σ_t^2 is weakly mixing. Is y_t weakly mixing?

Solution: When $\alpha = 0$, y_t is simply a 2-state Markov Swithcing Stochastic Volatility process. For this case, we have that σ_t^2 is weakly mixing, because S_t is. Moreover, from the lecture notes we have that y_t is weakly mixing because σ_t^2 is.

Question B.3: Suppose that S_t is i.i.d. with $P(S_t = 1) = p$ and $P(S_t = 2) = 1 - p$. State the density of y_t given y_{t-1} and $S_t = 1$. That is, find

$$f(y_t | y_{t-1}, S_t = 1). \tag{7}$$

Likewise, find $f(y_t | y_{t-1}, S_t = 2)$.

Solution:

$$f(y_t | y_{t-1}, S_t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{y_t^2}{2\sigma_t^2}\right) \quad \text{with } \sigma_t^2 = \omega_0 + \omega_1 1_{(S_t=1)} + \alpha y_{t-1}^2.$$

Hence

$$f(y_t | y_{t-1}, S_t = 1) = \frac{1}{\sqrt{2\pi(\omega_0 + \omega_1 + \alpha y_{t-1}^2)}} \exp\left(-\frac{y_t^2}{2(\omega_0 + \omega_1 + \alpha y_{t-1}^2)}\right)$$

and

$$f(y_t|y_{t-1}, S_t = 2) = \frac{1}{\sqrt{2\pi(\omega_0 + \alpha y_{t-1}^2)}} \exp\left(-\frac{y_t^2}{2(\omega_0 + \alpha y_{t-1}^2)}\right).$$

Question B.4: We want to estimate the model parameters $\theta = (\omega_0, \omega_1, \alpha, p)$ based on the EM algorithm. First, we seek to compute the EM log-likelihood function $L_{EM}(\theta)$ which we use in the expectation step (the E-step).
Treating $(S_t)_{t=1}^T$ as observed variables, consider the infeasible log-likelihood function defined as,

$$L(y_1, ..., y_T, S_1, ..., S_T; \theta) = \sum_{t=2}^{T} \left\{ 1_{(S_t=1)}[\log f(y_t|y_{t-1}, S_t = 1) + \log(p)] \right.$$
$$\left. +1_{(S_t=2)}[\log f(y_t|y_{t-1}, S_t = 2) + \log(1-p)] \right\}.$$

Recall that the E-step relies on making a guess of θ, $\theta = \tilde{\theta}$ say, and next computing

$$L_{EM}(\theta) = E_{\tilde{\theta}}[L(y_1, ..., y_T, S_1, ..., S_T; \theta)|y_1, ..., y_T].$$

This includes the computation of

$$P_t^*(1) := E_{\tilde{\theta}}[1_{(S_t=1)}|y_1, ..., y_T] = f_{\tilde{\theta}}(S_t = 1|y_1, ..., y_T),$$

where $f_{\tilde{\theta}}(S_t = 1|y_1, ..., y_T)$ denotes the probability (or density) $f(S_t = 1|y_1, ..., y_T)$ evaluated at $\tilde{\theta}$.

Show that, under the conditions in Question B.3 that for the case of $t = 2$,

$$f(S_2 = 1|y_1, y_2, ..., y_T) = \frac{f(y_2, ..., y_T|S_2 = 1, y_1)f(S_2 = 1, y_1)}{\sum_{i=1}^2 f(S_2 = i, y_1, ..., y_T)}.$$

Solution:

$$\begin{aligned} f(S_2 = 1|y_1, ..., y_T) &= \frac{f(S_2 = 1, y_1, ..., y_T)}{f(y_1, ..., y_T)} \\ &= \frac{f(y_2, ..., y_T|S_2 = 1, y_1)f(S_2 = 1, y_1)}{f(y_1, ..., y_T)} \\ &= \frac{f(y_2, ..., y_T|S_2 = 1, y_1)f(S_2 = 1, y_1)}{\sum_{i=1}^2 f(s_t = i, y_1, ..., y_T)}. \end{aligned}$$

Question B.5: Using the above, and with $P_t^*(2) = f_{\tilde{\theta}}(S_t = 2|y_1, ..., y_T)$ it follows that

$$L_{EM}(\theta) = \sum_{t=2}^{T} \{ P_t^*(1) [\log f(y_t|y_{t-1}, S_t = 1) + \log(p)]$$
$$+ P_t^*(2) [\log f(y_t|y_{t-1}, S_t = 2) + \log(1-p)] \} .$$

Explain how this EM-log-likelihood function can be used to find an estimate of θ.

Solution: Given $P_t^*(1)$ and $P_t^*(2)$, θ is estimated by maximizing $L_{EM}(\theta)$ over θ. One should relate this to the EM algorithm. The initial choice $\theta = \tilde{\theta}$ may not be good, and one can use the estimate of θ for the computation of new smoothed probabilities $P_t^*(1)$ and $P_t^*(2)$ in order to find a new estimate of θ. This procedure will typically be repeated "until convergence". The computation of $P_t^*(1)$ and $P_t^*(2)$ will typically be based on the forward and backward probabilities.